Field-line transport in stochastic magnetic fields: Percolation, Lévy flights, and non-Gaussian dynamics

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The transport of magnetic field lines is studied numerically in the case where strong three-dimensional magnetic fluctuations are superimposed to a uniform average magnetic field. The magnetic percolation of field lines between magnetic islands is found, as well as a non-Gaussian regime where the field lines exhibit Lévy random walks, changing from Lévy flights to trapped motion. Anomalous diffusion laws $\langle \Delta x_i^2 \rangle \propto s^\alpha$ with $\alpha > 1$ and $\alpha < 1$ are found for low fluctuation levels, while normal diffusion and Gaussian random walks are recovered for sufficiently high fluctuation levels.

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In this paper we report results on the transport of field lines in a stochastic magnetic field. Particle diffusion due to the random walk of field lines in the plane perpendicular to the mean magnetic field has been considered for a long time as the cause of loss of plasma confinement in fusion devices [1–3] and of fast perpendicular transport in many astrophysical phenomena [4]. In the magnetostatic approximation the field-line equations can be written as

$$\frac{d\mathbf{r}}{ds} = \frac{\mathbf{B}(\mathbf{r})}{|\mathbf{B}(\mathbf{r})|}, \tag{1}$$

where $\mathbf{B}(\mathbf{r})$ is the magnetic field at a generic point $\mathbf{r}$, and s is the field-line length. The magnetic field-line transport law, on which particle transport depends even in the case where collisions are important [2,5], can be obtained from the study of the above equations. While much numerical effort has been devoted to the study of particle transport in toroidal fusion devices [6], where the geometry is complicated but the number of wave modes involved is not too large, less attention has been given to the uniform average field case with a very large number of wave modes, which models astrophysical situations in the presence of strong magnetohydrodynamic turbulence.

Here we present the results of the simulation of a three-dimensional turbulent magnetic field in plane geometry, given by $\mathbf{B} = B_0\mathbf{e}_z + \delta\mathbf{B}$. The equation for the magnetic surfaces can be written as

$$\mathbf{B} \cdot \nabla \psi = 0, \tag{2}$$

where $\psi(x, y, z) = $ const denotes a magnetic surface. Provided that ψ is a bounded function—as it must be for a periodic magnetic field like that considered in the following—for every plane $z = z_0$ there will be a ψ contour, namely, the separatrix, that percolates all over the plane [7]. This can be viewed as the contour that separates the wells of ψ from the hills of ψ, so that the cross sections of the magnetic surfaces are composed by "magnetic islands" embedded in a web of separatrices. In the case that $\delta B \equiv \sqrt{\langle \delta\mathbf{B} \cdot \delta\mathbf{B} \rangle} \ll B_0$ the magnetic surfaces are nearly cylinders; further, in a periodic turbulent field the Kolmogorov-Arnold-Moser (KAM) theorem applies, and "good" (closed) magnetic surfaces exist. Increasing the perturbation level, some of the magnetic surfaces are "destroyed" because of stochasticity [1], and the thin regions next to the separatrices are the first to become stochastic, due to the presence of the hyperbolic points of $\psi(x, y, z_0)$. Therefore magnetic field-line transport is concentrated on thin sheaths surrounding good magnetic surfaces (magnetic percolation). The magnetic field-line equations correspond to a 1.5 degrees of freedom Hamiltonian system and thus should have the properties of these systems. As long as $\delta B \ll B_0$, the measure of the stochastic region will be small. In such a situation of weak chaos, transport can be inherently non-Gaussian, i.e., anomalous, in the sense that $\langle \Delta x_i^2 \rangle \propto s^\alpha$, with $\alpha \neq 1$ [8] (here and in the following, $\Delta x_i = x_i - \langle x_i \rangle$, and $x_1 = x$, $x_2 = y$). As the fluctuation amplitude is further increased, the good magnetic surfaces are progressively destroyed, and global stochasticity is attained. Therefore a transition from an anomalous regime to a normal diffusion regime where $\langle \Delta x_i^2 \rangle \propto s$ can be expected.

For the numerical simulation, we consider a fluctuating magnetic field given by

$$\delta\mathbf{B}(\mathbf{r}) = \sum_{\mathbf{k},\sigma} \delta B_\sigma(\mathbf{k}) \mathbf{e}_\sigma(\mathbf{k}) \exp[i(\mathbf{k} \cdot \mathbf{r} + \varphi_{\mathbf{k}}^\sigma)], \tag{3}$$

where $\delta B_\sigma(\mathbf{k})$ is the Fourier amplitude of the mode with wave vector $\mathbf{k}$ and polarization σ, $\mathbf{e}_\sigma(\mathbf{k})$ are the polarization unit vectors, $\nabla \cdot \mathbf{B} = 0$ implies $\mathbf{k} \cdot \mathbf{e}_\sigma(\mathbf{k}) = 0$, and $\varphi_{\mathbf{k}}^\sigma$ are random phases, chosen to simulate a realization of an ensemble of stochastic magnetic fields. The basic periodicity is that of the cubic simulation box of side L, so that $\mathbf{k} = 2\pi(n_x, n_y, n_z)/L$. The Fourier amplitudes for an isotropic power law spectrum with correlation length λ are given by [9]

$$\delta B_\sigma(\mathbf{k}) = C(2\pi/L)^{3/2}/(k^2\lambda^2 + 1)^{\gamma/4+1/2}, \tag{4}$$

where C is a normalization constant, k the wave number, and γ the spectral index. For the present calculations $\lambda = L$, while the spectral index is fixed as $\gamma = 3/2$, which corresponds to the Kraichnan spectrum for fully

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developed magnetohydrodynamic turbulence [10]. The spectrum is truncated at $k_{\max} = 2\pi N/L$, where $N = L/\lambda_{\min}$, with $\lambda_{\min}$ the smallest turbulence wavelength present, and the fluctuation amplitude is defined as $A = \delta B/B_0$. At each integration step, the sum of $\sim 4N^3$ components should be evaluated many times. In order to save computer time, we introduce a three-dimensional (3D) lattice with grid points spaced by $\lambda_{\min}/10$, on which the magnetic field components are computed exactly [11]. Then, during the integration of Eq. (1), the magnetic field at a generic point is obtained by quadratic interpolation of the ten nearest grid points. A Runge-Kutta sixth order integrator in double precision is used.

Extensive simulations have been performed for $N = 3, 6, 9$, and 12, corresponding to 61, 462, 1535, and 3576 wave vectors, respectively, each having two independent polarizations. Since the simulated field is periodic, the onset of stochasticity can be studied by means of the Poincaré surfaces of section obtained by drawing for a given field line a point in the (x,y) plane for each integer value of z/L. For $A \ll 1$ magnetic islands are found, separated from each other by percolating separatrices. Increasing A, the field-line motion is more and more chaotic, until the last KAM torus is destroyed. Nevertheless, field-line transport is not yet statistically homogeneous: as shown for $N = 12$ and $A = 0.30$ in Fig. 1, field lines propagate preferentially along the remnants of the separatrices. This means that even in a regime of global stochasticity field lines have memory of the magnetic configuration obtained for lower fluctuation levels. Also, field lines can be trapped for some time in the stochastic region corresponding to a former magnetic island, and then travel long paths, which are called "Lévy flights" [12–15]. This is elucidated in Fig. 2, where a sample field line projected in the xy plane is plotted. The trapping and subsequent Lévy flight of field lines is apparent. In the case of Lévy flights transport is superdiffusive, the limit distribution of field lines is not a Gaussian but one of the Lévy stable laws, and the set of points of the Poincaré section of a field line is a fractal [14]. Note that increasing A diminishes the probability of long flights and of long trapping, and therefore diminishes the average length of percolation in the stochastic layer.

A quantitative study of magnetic field-line transport is carried out by computing $\langle \Delta x_i^2 \rangle$ as a function of s. These averages are computed over an ensemble of 2000 lines of force integrated up to either $s = 500$ or $s = 1000$ (all the lengths are normalized to L). The integration length has been chosen so as to attain the long time behavior of the random walk, checking that asymptotic values are obtained. The transport law is found to be

$$\langle \Delta x_i^2 \rangle = 2 D_i s^{\alpha_i}, \qquad s \to \infty. \tag{5}$$

Here, α_i characterizes the random walk law: $\alpha_i = 1$ in the diffusion regime, $\alpha_i = 2$ in the ballistic regime, and $1 < \alpha_i < 2$ in the case of Lévy random walk [13,14,8]. In the case of trapping $\alpha_i < 1$ (subdiffusive regime). The results for α_x and α_y as a function of A are reported in Fig. 3 for the various N. Both superdiffusive and subdiffusive behavior is found for the motion in the xy plane for small values of A, and a diffusion regime is reached for larger values of A. Because of the increase of stochasticity of field lines with A, the probability of going from a "trapped" to a "percolating" path or vice versa increases, see Fig. 2, so that the field-line motion approaches a Gaussian random walk with a finite scale length and the ordinary diffusion behavior is recovered. Note that for low A the character of transport is different in the x and y directions. This is due to the fact that usually the percolating field lines go from one side to the opposite of the simulation box, and keep moving in the same direction because of the periodicity of $\delta\mathbf{B}$.

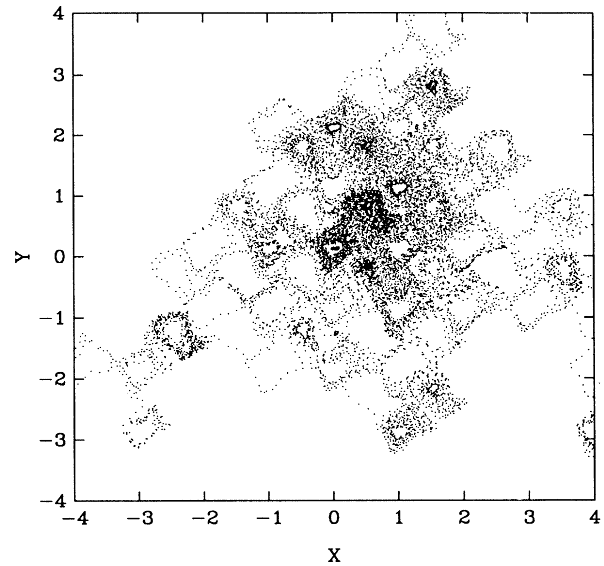

FIG. 1. Poincaré surface of section for $N = 12$ and $A = 0.30$. Thirty field lines are plotted for 500 integer values of z. Dimensionless units.

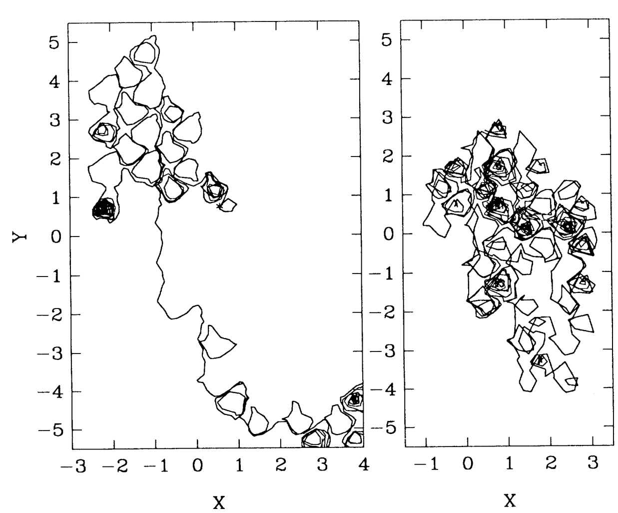

FIG. 2. Projection on the xy plane of a field line with given initial conditions (indicated by an asterisk) for $N = 9$. Dimensionless units. Left: $A = 0.45$. Right: $A = 0.70$.

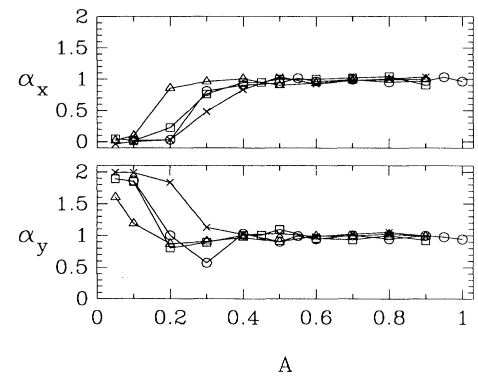

FIG. 3. The anomalous diffusion exponents α_x and α_y versus the fluctuation amplitude A. Dimensionless units. Circles: $N = 3$; crosses: $N = 6$; squares: $N = 9$; triangles: $N = 12$.

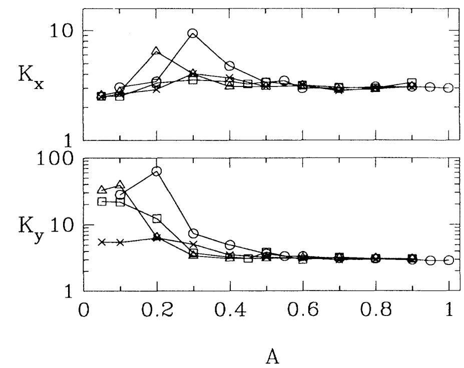

FIG. 4. The kurtoses K_x and K_y versus A (same symbols as in Fig. 3). Dimensionless units.

On the other hand, this anisotropy can be flipped over by changing the random phases, as has been verified.

The Lévy random walk corresponds to a "broad" probability distribution $p(l)$ of making a step of length l in a given time interval of the form $p(l) \sim l^{-(\mu+1)}$ for $l \to \infty$ [14]. The cases with $\mu > 2$ correspond to a Gaussian random walk law, whereas $\mu < 2$ gives rise to a random walk law of the form of Eq. (5), with $\alpha = 2/\mu$ [14] [in such a case, the second order moment of $p(l)$—the mean square jump length—is infinite]. Therefore we can estimate the values of the exponent μ of the probability distribution $p(l)$ by means of the numerically obtained values of α. Note that when $\alpha > 1$, μ corresponds to the fractal dimension of the points constituting the random walk [13]. On the other hand, when $\alpha < 1$ the microscopic dynamics evolves on a fractal time and the random walk is intermittent [15] (in the present case, s is the timelike parameter). This can be described by a "broad" probability distribution $q(s)$ of waiting "times" s between two disentangled steps of the form $q(s) \sim s^{-(\nu+1)}$ for $s \to \infty$, and $\alpha = \nu$ results for $\nu < 1$ [14,15].

An indicator of non-Gaussian dynamics is the kurtosis of the distribution function of field lines, defined as $K_i = \langle \Delta x_i^4 \rangle / \langle \Delta x_i^2 \rangle^2$. For a Gaussian distribution function $K_i = 3$, while $K_i > 3$ ($K_i < 3$) corresponds to enhanced (reduced) importance of the tails of the distribution. We have computed K_x and K_y for all the magnetic configurations studied, and the results at the end of the integration time are displayed in Fig. 4. As can be seen, large departures from $K_i = 3$ are found for low to moderate values of A, but the kurtoses tend to the Gaussian value roughly when the corresponding anomalous diffusion exponent α_i tends to one. For the x direction, where trapping of field lines prevails at low A, we have $K_x < 3$ for $A = 0.05$, but increasing the fluctuation amplitude K_x surpasses 3 before settling to the Gaussian value. This means that in the regime where several magnetic surfaces are destroyed some field lines experience Lévy flights in the x direction, too, increasing the relevance of the tails of the distribution, even if the overall behavior is subdiffusive because most field lines are trapped.

We consider that we are in the Gaussian regime when both $|\alpha_i - 1| < 0.1$ and $|K_i - 3| < 0.3$ (a 10% error criterion). In such a case we can compare the coefficients D_i of Eq. (5) to the quasilinear diffusion coefficient obtained from Eq. (4) for $k_{\max} \to \infty$,

$$D_{QL} = \frac{\sqrt{\pi}}{8} \frac{\gamma - 1}{\gamma} \frac{\Gamma(\gamma/2 + 1)}{\Gamma(\gamma/2 + 1/2)} \lambda \left(\frac{\delta B}{B_0} \right)^2, \qquad (6)$$

where Γ is Euler's gamma function, and to the recently proposed percolation scaling $D_m \propto (\delta B/B_0)^{7/10}$ [5]. D_x, D_y, and $D_\perp = D_x + D_y$ are reported in Fig. 5. First of all, note that for both the x and y directions the Gaussian regime is reached the sooner, the higher N. This is best seen in the lower panel of Fig. 5, where $D_\perp$ is plotted, and matches with the fact that global stochasticity is attained earlier with a longer spectrum, since more hyperbolic points and larger gradients of $\mathbf{B}$ are present. Indeed, the rate of exponential separation

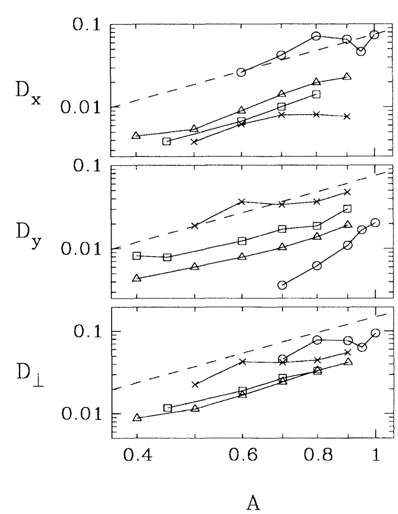

FIG. 5. Computed perpendicular diffusion coefficients (same symbols as in Fig. 3) compared to the quasilinear values (dashed line). Dimensionless units.

of field lines depends on the gradient of **B** [2]. Transport appears to be markedly anisotropic even well into the Gaussian regime, especially for $N = 3$ and $N = 6$. For the magnetic configuration obtained with $N = 3$ the anisotropy changes from prevalently along the y axis in the non-Gaussian regime (low A) to prevalently along the x axis in the Gaussian regime, a change which is clearly seen also in the Poincaré sections and in the values of $\langle \Delta x^2 \rangle$ and $\langle \Delta y^2 \rangle$. Although this irregularity could be attributed to the particular realization of the ensemble of magnetic fields which we have simulated, these results indicate that field-line transport is quite complex, depending on the deep dynamics of field lines in the remnants of separatrices and magnetic islands [8]. More regularity is found in $D_\perp$ as a function of N and of A, since the anisotropy effects are averaged out. Note in particular that (i) $D_\perp$ is always smaller than the quasilinear diffusion coefficient, as it should be when $A \sim 1$; (ii) at a given fluctuating energy density level, $D_\perp$ is larger for a smaller N, in agreement with the fact that more energy is found at longer wavelengths and that $D_{QL} \sim \lambda(\delta B/B_0)^2$ [3]; (iii) for $N = 9$ and $N = 12$ transport is nearly isotropic and the values of $D_\perp$ are pretty close, a fact which indicates that we are near to the saturation of $D_\perp$ as a function of the number of wave modes.

Inspection of the lower panel of Fig. 5 also shows that the scaling of $D_\perp$ with the fluctuation amplitude is more similar to the quasilinear than to the percolation one. The $(\delta B/B_0)^{7/10}$ scaling was obtained by Isichenko [5] for a monoscale turbulence in the case $\delta B/B_0 \ll 1$ and $\lambda_\parallel/\lambda_\perp \gg 1$ (here, $\lambda_\parallel$ and $\lambda_\perp$ are the parallel and transverse correlation lengths, respectively), whereas we have $\delta B/B_0 \sim 1$ and an isotropic spectrum for $\delta\mathbf{B}$. This can explain why Isichenko's scaling is not reproduced.

In conclusion, our simulation of stochastic magnetic fields shows that for low fluctuation levels magnetic field-line transport is non-Gaussian and the random walk is composed of trapping in the remnants of the KAM tori and of long ballistic flights (Lévy flights) in the percolation layer. The transport law is anomalous, $\langle \Delta x_i^2 \rangle \propto 2D_i s^{\alpha_i}$ with $\alpha > 1$ and $\alpha < 1$. On the other hand, for higher fluctuation levels a Gaussian diffusion regime is attained; however, the perpendicular diffusion coefficient is smaller than the quasilinear value, and transport may be rather anisotropic.

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